Metody dowodzenia twierdzeń
i automatyzacja rozumowań
Dowody dynamiczne w logikach adaptatywnych

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DISCLAIMER
These are just notes, do not overrate them.

The story comes from:

- Diderik Batens, “The Need for Adaptive Logics in Epistemology”
- Joke Meheus, “Adaptive Logics and the Integration of Induction and Deduction”
- Joke Meheus, “Clausius’ Discovery of the First Two Laws of Thermodynamics. A Paradigm of Reasoning from Inconsistencies”
Dynamics of reasoning – external vs internal

- **External dynamics** – non-monotonic reasoning.
  
  Easy to simulate even with monotonic inference relations – just add some constraints on derivability (like consistency, minimality and so on).

- **Internal dynamics** – conclusions are provisional, even if the set of premises is stable.
  
  Modeling this kind of dynamics requires specific inference mechanisms to be defined.
Consider the case in which a theory $\mathcal{T}$ was meant to be consistent and was formulated with classical logic (CL) as its underlying logic, but turned out to be inconsistent. [...] scientists do not just throw away such theory. They reason from $\mathcal{T}$ in search for a consistent replacement. [...] In their reasoning they want to interpret $\mathcal{T}$ as consistently as possible.
Example: handling inconsistencies [Batens 2004]

\[ \mathcal{T} = \{ \neg p, p \lor r, \neg q, q \lor s, p \} \]

In \( \mathcal{T} \) q behaves consistently, whereas p does not. Thus one shouldn’t apply rules like Disjunctive Syllogism* to p, but it is safe to apply it to q.

*From \( A \lor B, \neg A \) derive \( B \).
Adaptive logics [Meheus 2003]

What all these logics have in common is that, as their name implies, they adapt themselves to the specific set of premises to which they are applied.

This means that, at least for some of the inference rules, it depends on properties of the premise set whether their application is validated or not. Importantly, this does not require any intervention of the user – the logics really adapt themselves to the premises.

Moreover, as we shall see below, this adaptation proceeds in a contextual way – even if the application of an inference rule is invalidated with respect to some ‘parts’ of the premises, it may be validated with respect to others.
This contextual validation of inference rules is related to the application context for which adaptive logics are meant, namely reasoning processes that exhibit an external and/or internal dynamics.

A typical characteristic of such processes is that a specific set of presuppositions is maintained ‘as much as possible’, this is, unless and until they are explicitly violated.

For instance, it is not possible to reason sensibly from an inconsistent set of statements while presupposing that $A \land \neg A$ is false, for any possible $A$. Still, one may maintain this presupposition for all $A$ that have not explicitly been shown to behave inconsistently. If this presupposition is violated, because some $B$ has been found to behave inconsistently, then previously derived conclusions may be withdrawn.
Adaptive logics – formal characteristics

An adaptive logic may be characterized in terms of:

1. a lower limit logic (LLL – defines the rules of inference that hold unconditionally);
2. a set of abnormalities (the set of abnormalities \( \Omega \) consists of the formulas that are presupposed to be false, unless and until proven otherwise);
   - upper limit logic (ULL) – LLL + requirement ‘no abnormality is logically possible’; ULL characterizes the ‘normal’ situation, where no formula behaves abnormally; defines conditional inference rules;
3. an adaptive strategy.
Abnormalities – Inconsistency-adaptive logics

Abnormalities: formulas of the form $A \land \neg A$
(or, in the predicative case, their existential closures)

Application context of these logics concerns theories that were intended to be interpreted in terms of CL, but that were discovered to be inconsistent. Applying IALs to such a theory ensures that the theory is interpreted as consistently as possible: one obtains the theory in its full richness, except for the pernicious consequences of its inconsistency.
Abnormalities: formulas of the form $\exists A \land \exists \neg A$
(where $A$ is purely functional – no individual constants, no quantifiers)

This system is intended to be applied to sets of premises that consist, on the one hand, of a number of empirical data (singular statements) and, on the other hand, of a number of background generalizations. From these, it leads to a number of new generalizations (that are jointly compatible with the premises) and to the CL-consequences of these new generalizations, the data and the background generalizations.

The basic idea is to assume (unless and until proven false) that, if something holds true for some objects, it holds true for all objects.
Abnormalities: formulas of the form
\[ \forall x (A(x) \rightarrow B(x)) \land \neg (B(x/\alpha) \rightarrow A(x/\alpha)) \]

The idea behind any adaptive logic for abduction is that it should lead from a set of explananda together with a background theory to a set of explanatory hypotheses as well as to the CL-consequences of these and the background theory.

Intended application context of this logic is the situation in which explanatory hypotheses are derived as a basis for actions. In line with this, it leads to a set of explanatory hypotheses that are jointly compatible with the premises and that are moreover as weak as possible in view of them.
Abnormalities

- $\Omega$ – the set of all abnormalities (relative to a logic)
- $Dab(\Delta)$ – disjunction of abnormalities ($\Delta$ is a finite subset of $\Omega$);
- $Dab(\Delta)$ is a $Dab$-consequence of a set $X$ iff $Dab(\Delta)$ is derivable from $X$ by LLL;
- $Dab(\Delta)$ is a minimal $Dab$-consequence of a set $X$ iff it is a $Dab$-consequence of $X$ and there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ a $Dab$-consequence of $X$.

If $Dab(\Delta)$ is a minimal $Dab$-consequence of $X$, then one of the members of $\Delta$ behaves abnormally with respect to $X$, but it cannot be established which one. Hence, the phrase that $X$ should be interpreted “as normally as possible” (that is, that it should be assumed for as many abnormalities as possible that they are false) becomes ambiguous. It is in view of this complication that the third element, the adaptive strategy, is needed.
Basic adaptive strategies

- Reliability Strategy: a formula behaves abnormally with respect to $X$ if and only if it is a member of a minimal $Dab$-consequence of $X$.
- Minimal Abnormality Strategy: if $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal $Dab$-consequences of $X$, then the Minimal Abnormality Strategy first selects one member of each $\Delta_i$ to behave abnormally, and next, considers all such combinations.
Adaptive strategies – example: inconsistency-adaptive logics

\[ X = \{ \neg p, \neg q, p \lor q, p \lor r, q \lor r, s, \neg s \} \]

How to derive \( r \)?
Adaptive strategies – example: inconsistency-adaptive logics

\[ X = \{ \neg p, \neg q, p \lor q, p \lor r, q \lor r, s, \neg s \} \]

Abnormalities are of the form \( A \land \neg A \)

- **Reliability Strategy:** \( r \) is not an adaptive consequence of \( X \) (\( p, q, s \) are all considered as behaving abnormally).
- **Minimal Abnormality Strategy:** \( r \) is an adaptive consequence of \( X \) (either \( p, s \) or \( q, s \) behave abnormally).
Proof theory

Derivability Adjustment Theorem

\[ B_1, \ldots B_n \vdash_{ULL} A \text{ iff there is a finite } \Delta \subseteq \Omega \text{ such that} \]

\[ B_1, \ldots B_n \vdash_{LLL} A \lor Dab(\Delta) \]

- Rules: premise rule, unconditional rule, conditional rule;
- Marking definition

Five elements of proof line:

1. a line number,
2. the derived formula \( A \),
3. the line numbers of formulas from which \( A \) is derived,
4. the rule by which \( A \) is derived,
5. the condition.
If $A \in X$, then one may add to the proof a line consisting of:

1. the appropriate line number,
2. $A$,
3. “–”
4. “PREM”
5. $\emptyset$. 
RU – unconditional rule

If \( B_1, \ldots B_n \vdash_{LLL} A \ (n \geq 0) \) and \( B_1, \ldots B_n \) occur in the proof on the conditions \( \Delta_1, \ldots \Delta_n \) respectively, then one may add to the proof a line consisting of

1. the appropriate line number,
2. \( A \),
3. the line numbers of the \( B_i \),
4. "RU"
5. \( \Delta_1 \cup \ldots \cup \Delta_n \).
RC – conditional rule

If $B_1, \ldots B_n \vdash_{LLL} A \lor Dab(\Theta)$ ($n \geq 0$) and $B_1, \ldots B_n$ occur in the proof on the conditions $\Delta_1, \ldots \Delta_n$ respectively, then one may add to the proof a line consisting of

1. the appropriate line number,
2. $A$,
3. the line numbers of the $B_i$,
4. “RC”
5. $\Theta \cup \Delta_1 \cup \ldots \cup \Delta_n$. 
Reliability Strategy leads to the marking of a line (at some stage) if one of the elements in its condition is unreliable at that stage – that is, is a disjunct of a minimal $Dab$-formula that has been derived at that stage.

Let $U_s(X)$ be the union of all $\Delta$ for which $Dab(\Delta)$ is a minimal $Dab$-formula derived at stage $s$.

### Marking Definition

Line $i$ is marked at stage $s$ iff, where $\Delta$ is its fifth element, $\Delta \cap U_s(X) \neq \emptyset$. 
Derivability

A formula $A$ is derived at stage $s$ in a proof from $X$ iff $A$ is the second element of a line that is not marked in the proof (at stage $s$).

Final Derivability

$A$ is finally derived on line $i$ of a proof from $X$ iff (a) $A$ is the second element of line $i$, (b) line $i$ is not marked in the proof, and (c) any extension of the proof in which line $i$ is marked may be further extended in such a way that line $i$ is unmarked.

Proof Invariance

If $X \vdash A$, then any proof from $X$ can be extended into a proof in which $A$ is finally derived from $X$. 
Example: ACLuN1

- ULL – CL
- LLL – paraconsistent logic CLuN (full positive CL + \( p \lor \neg p \))

![Diagram]

- abnormalities: \( A \land \neg A \)
- Reliability Strategy
Derivability Adjustment Theorem in action

Note: for every consistent set \( X \), \( \text{Cn}_{ACLuN1}(X) = \text{Cn}_{CL}(X) \).

1. \( X_1 = \{ \neg t, t \lor s \} \)
   \( X_1 \vdash_{CL} s \)
   \( X_1 \vdash_{ACLuN1} s \lor (t \land \neg t) \)

2. \( X_2 = \{ u \rightarrow p, u \rightarrow \neg p \} \)
   \( X_2 \vdash_{CL} \neg u \)
   \( X_2 \vdash_{ACLuN1} \neg u \lor (p \land \neg p) \)

3. \( X_3 = \{ p, \neg p \lor q, r \rightarrow \neg q \} \)
   \( X_3 \vdash_{CL} \neg r \)
   \( X_3 \vdash_{ACLuN1} \neg r \lor ((p \land \neg p) \lor (q \land \neg q)) \)

4. \( X_4 = \{ \neg s, r \rightarrow s \} \)
   \( X_4 \vdash_{CL} \neg r \)
   \( X_4 \vdash_{ACLuN1} \neg r \lor (s \land \neg s) \)
Proof example

\[ X = \{ p \land \neg t, q \land (t \lor s), \neg p \lor \neg q, r \rightarrow \neg s, u \rightarrow p, u \rightarrow \neg p \} \]

1. \[ p \land \neg t \quad \text{– PREM} \quad \emptyset \]
2. \[ q \land (t \lor s) \quad \text{– PREM} \quad \emptyset \]
3. \[ \neg p \lor \neg q \quad \text{– PREM} \quad \emptyset \]
4. \[ r \rightarrow \neg s \quad \text{– PREM} \quad \emptyset \]
5. \[ u \rightarrow p \quad \text{– PREM} \quad \emptyset \]
6. \[ u \rightarrow \neg p \quad \text{– PREM} \quad \emptyset \]
Proof example

1. $p \land \neg t$ – PREM $\emptyset$
2. $q \land (t \lor s)$ – PREM $\emptyset$
3. $\neg p \lor \neg q$ – PREM $\emptyset$
4. $r \rightarrow \neg s$ – PREM $\emptyset$
5. $u \rightarrow p$ – PREM $\emptyset$
6. $u \rightarrow \neg p$ – PREM $\emptyset$
7. $\neg u$ 5, 6 RC $\{p \land \neg p\}$
### Proof Example

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Reason</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p \land \neg t$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>$q \land (t \lor s)$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3</td>
<td>$\neg p \lor \neg q$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>4</td>
<td>$r \rightarrow \neg s$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>5</td>
<td>$u \rightarrow p$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>6</td>
<td>$u \rightarrow \neg p$</td>
<td>PREM</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>7</td>
<td>$\neg u$</td>
<td>RC</td>
<td>${p \land \neg p}$</td>
</tr>
<tr>
<td>8</td>
<td>$\neg t$</td>
<td>RU</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>9</td>
<td>$t \lor s$</td>
<td>RU</td>
<td>$\emptyset$</td>
</tr>
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</table>
Proof example

1. \( p \land \neg t \) – PREM \( \emptyset \)
2. \( q \land (t \lor s) \) – PREM \( \emptyset \)
3. \( \neg p \lor \neg q \) – PREM \( \emptyset \)
4. \( r \rightarrow \neg s \) – PREM \( \emptyset \)
5. \( u \rightarrow p \) – PREM \( \emptyset \)
6. \( u \rightarrow \neg p \) – PREM \( \emptyset \)
7. \( \neg u \) 5, 6 RC \( \{ p \land \neg p \} \)
8. \( \neg t \) 1 RU \( \emptyset \)
9. \( t \lor s \) 2 RU \( \emptyset \)
10. \( s \) 8, 9 RC \( \{ t \land \neg t \} \)
Proof example

1. \( p \land \neg t \) – PREM \( \emptyset \)
2. \( q \land (t \lor s) \) – PREM \( \emptyset \)
3. \( \neg p \lor \neg q \) – PREM \( \emptyset \)
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8. \( \neg t \) 1 RU \( \emptyset \)
9. \( t \lor s \) 2 RU \( \emptyset \)
10. \( s \) 8, 9 RC \( \{ t \land \neg t \} \)
11. \( \neg r \) 4, 10 RC \( \{ t \land \neg t, s \land \neg s \} \)
Proof example

1. \( p \land \neg t \) – PREM \( \emptyset \)
2. \( q \land (t \lor s) \) – PREM \( \emptyset \)
3. \( \neg p \lor \neg q \) – PREM \( \emptyset \)
4. \( r \rightarrow \neg s \) – PREM \( \emptyset \)
5. \( u \rightarrow p \) – PREM \( \emptyset \)
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8. \( \neg t \) 1 RU \( \emptyset \)
9. \( t \lor s \) 2 RU \( \emptyset \)
10. \( s \) 8, 9 RC \( \{t \land \neg t\} \)
11. \( \neg r \) 4, 10 RC \( \{t \land \neg t, s \land \neg s\} \)
12. \((p \land \neg p) \lor (q \land \neg q)\) 1, 2, 3 RU \( \emptyset \)

12 is a \textit{Dab}-formula. According to Reliability Strategy, both \( p \land \neg p \) and \( q \land \neg q \) are unreliable. According to Marking Definition all lines whose fifth element (a condition) contains \( p \land \neg p \) or \( q \land \neg q \) should be marked. Thus, line 7 is to be marked.
### Proof example

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Line(s)</th>
<th>Type</th>
<th>Marking</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p \land \neg t$</td>
<td></td>
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</tr>
<tr>
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<td>1</td>
<td>RU</td>
<td>$\emptyset$</td>
</tr>
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<td>9</td>
<td>$t \lor s$</td>
<td>2</td>
<td>RU</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>10</td>
<td>$s$</td>
<td>8, 9</td>
<td>RC</td>
<td>${t \land \neg t}$</td>
</tr>
<tr>
<td>11</td>
<td>$\neg r$</td>
<td>4, 10</td>
<td>RC</td>
<td>${t \land \neg t, s \land \neg s}$</td>
</tr>
<tr>
<td>12</td>
<td>$(p \land \neg p) \lor (q \land \neg q)$</td>
<td>1, 2, 3</td>
<td>RU</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

12 is a *Dab*-formula. According to Reliability Strategy, both $p \land \neg p$ and $q \land \neg q$ are unreliable. According to Marking Definition all lines whose fifth element (a condition) contains $p \land \neg p$ or $q \land \neg q$ should be marked. Thus, line 7 is to be marked.
Proof example

1. \( p \land \neg t \) – PREM \( \emptyset \)
2. \( q \land (t \lor s) \) – PREM \( \emptyset \)
3. \( \neg p \lor \neg q \) – PREM \( \emptyset \)
4. \( r \rightarrow \neg s \) – PREM \( \emptyset \)
5. \( u \rightarrow p \) – PREM \( \emptyset \)
6. \( u \rightarrow \neg p \) – PREM \( \emptyset \)
7. \( \neg u \) 5, 6 RC \( \{ p \land \neg p \} \) \( \blackleftarrow_{12} \)
8. \( \neg t \) 1 RU \( \emptyset \)
9. \( t \lor s \) 2 RU \( \emptyset \)
10. \( s \) 8, 9 RC \( \{ t \land \neg t \} \)
11. \( \neg r \) 4, 10 RC \( \{ t \land \neg t, s \land \neg s \} \)
12. \( (p \land \neg p) \lor (q \land \neg q) \) 1, 2, 3 RU \( \emptyset \)

At stage 12, the formula on line 7 is no longer considered as derived. It is easily observed that 7 will remain marked and hence that \( \neg u \) is not finally derivable from the premises (whereas, e.g. 10 and 11 are finally derived).
Some conclusions

- A unified framework for studying a variety of reasoning processes (reasoning from inconsistencies, inductive generalizations, abduction, analogical reasoning,...)
- Possibility of reconstruction (in a realistic and formally exact way) actual cases of scientific reasoning (cf. Meheus [1999]).
- Better insight in the appraisal of dynamic arguments (their soundness, in particular)
- The role of empirical constraints:

Logicians either design systems that can only be applied by gods and angels or they take into account some of the characteristics of actual reasoners. [...] the problems concerning scientific reasoning seem too important and too interesting not to take the second road.

Questions:

1. Logic or heuristics? Normative impact?
2. Decent criteria for final derivability?
Bibliography


- Meheus, J. [1999] Clausius’ Discovery of the First Two Laws of Thermodynamics. A Paradigm of Reasoning from Inconsistencies, Philosophica, 63, s. 89–117